On Optimality of Beamforming for Multiple Antenna Systems with Imperfect Feedback

Syed Ali Jafar, Andrea Goldsmith
Department of Electrical Engineering
Stanford University, Stanford, CA 94305, USA
{*syed, andrea*}*@wsl.stanford.edu*

Abstract

We determine a necessary and sufficient condition under which beamforming achieves Shannon capacity in a narrowband point to point communication system employing multiple transmit antennas and a single receive antenna. We assume perfect channel state information at the receiver (CSIR) and imperfect channel state feedback from the receiver to the transmitter. We consider the cases of mean and covariance feedback. The channel is modeled at the transmitter as a vector of complex jointly Gaussian random variables with either a zero mean and a known covariance matrix (covariance feedback), or a non-zero mean and a white covariance matrix (mean feedback). For both cases we develop a necessary and sufficient condition for when the Shannon capacity is achieved through beamforming, i.e. with a unit rank input covariance matrix. We also provide a waterpouring interpretation of our results and find that better feedback quality not only increases the system capacity but does so with scalar coding which involves significantly less complexity than vector coding.

I. Introduction

Recently there has been much interest in the capacity of multiple antenna systems with partial channel state information at the transmitter (CSIT). It has been found that unlike single antenna systems where exploiting CSIT does not significantly enhance the Shannon capacity, for multiple antenna systems the capacity improvement through even partial CSIT can be substantial. Key work on capacity of such systems by Telatar [1], Visotsky [2] and Narula [3] has provided some interesting results and lead to new questions. For a given input covariance matrix the input distribution that achieves the Shannon capacity is shown in [1] to be complex vector Gaussian, mainly because the vector Gaussian distribution maximizes the entropy for a given covariance matrix. This leads to the capacity optimization problem - i.e., finding the optimum input covariance matrix to maximize capacity subject to a transmit power (trace of input covariance matrix) constraint. The optimum covariance matrix in general can be a full rank matrix which implies vector coding across the antenna array. Limiting the rank of the input covariance matrix to unity (beamforming) essentially leads to a scalar coded system which has a significantly lower complexity for typical array sizes.

The capacity optimization problem was recently solved by Visotsky and Madhow in [2] for the cases of mean feedback and covariance feedback. Moreover, their numerical results indicate that beamforming is close to the optimal strategy when the quality of feedback improves (mean feedback) or a stronger path is present (covariance feedback). For *mean feedback*, Narula and Trott [3] point out that there are cases where the capacity is actually achieved via beamforming. While they do not obtain fully general necessary and sufficient conditions for when beamforming is a capacity achieving strategy, they develop partial answers to the problem for two transmit antennas. Note that both [3] and [2] assume multiple transmit antennas and a single receive antenna.

In this work we develop the general necessary and sufficient condition for optimality of beamforming for systems using multiple transmit antennas and a single receive antenna under both mean and covariance feedback.

II. System Model and Problem Statement

We use the same system model and notation as [2]. We focus on a point-to-point communication system using n_T transmit antennas and a single receive antenna over a narrowband flat fading channel. It is assumed that at each time instant the exact values of the channel coefficients, represented by the column vector $\mathbf{h}$, are available to the receiver while the transmitter, based on imperfect feedback [2], models the channel as a random complex Gaussian vector $\mathbf{h} \sim \tilde{N}(\mu, \Sigma)$ (Rayleigh Fading). The capacity can be shown to be [2]

$$C = \max_{Q \,:\, \mathrm{trace}(Q)=P} \mathrm{E}\left[\log[1 + \frac{\mathbf{h}^\dagger Q \mathbf{h}}{\sigma^2}]\right] \tag{1}$$

where the expectation is over the channel state $\mathbf{h}$. The capacity is achieved by a Gaussian codebook which is specified completely by the optimum input covariance matrix Q^o. The optimum transmit strategy is to transmit independent complex circular Gaussian symbols along the eigenvectors of Q^o. The powers allocated to each eigenvector are given by the eigenvalues of Q^o. Thus the transmit power constraint is given as trace(Q^o)= P.

Consistent with [2] and [4], we define beamforming as a transmission strategy where the input covariance matrix Q has rank one. As pointed out in [4], this definition of beamforming is different from the interpretation it has in directive array literature where there is no stochastic model for the channel parameters and perfect CSIT is assumed. We solve the following problem.

Problem Statement For the case of mean feedback, where $\mathbf{h} \sim \tilde{N}(\mu, \alpha\mathbf{I})$, and for the case of covariance feedback, where $\mathbf{h} \sim \tilde{N}(\mathbf{0}, \Sigma)$, determine a necessary and sufficient condition for optimality of beamforming. In other words we wish to determine a necessary and sufficient condition on μ, α, Σ, P and σ^2 for the optimal covariance matrix (that maximizes (1)) to have unit rank.

Note that the channel distribution (α, μ, Σ) may change over time. However, since the channel distribution changes much more slowly than the channel itself, we assume local stationarity and are interested in the capacity C for each given, fixed channel distribution. In general one can define a stochastic model for the parameters α, μ and Σ and adapt the transmit power P to the channel distribution. Once the optimum power adaptation is determined, this would still require solving the problem stated earlier for each $P(\alpha, \mu)$ or $P(\Sigma)$. Moreover numerical results in [5] show that adapting the transmit power to the feedback results in little performance gain as compared to keeping the transmit power constant.

III. Previous Results

The solutions to the capacity optimization problem and the previous results on optimality of beamforming presented in [2] and [3] are summarized here.

A. Capacity Optimization - Covariance Feedback

For covariance feedback $\mathbf{h} \sim \tilde{N}(\mathbf{0}, \Sigma)$. Then, as shown in [2], the maximizing covariance matrix Q^o and the channel covariance matrix Σ have the same eigenvectors. So the spectral decompositions can be expressed as $Q^o = U_\Sigma \Lambda_Q^o U_\Sigma^\dagger$ and $\Sigma = U_\Sigma \Lambda_\Sigma U_\Sigma^\dagger$. The optimal strategy is to employ independent complex circular Gaussian inputs along the eigenvectors of Σ. Λ_Q^o is a diagonal matrix whose elements λ_i^Q need to be determined through numerical maximization techniques subject to the trace constraint.

B. Capacity Optimization - Mean Feedback

For mean feedback $\mathbf{h} \sim \tilde{N}(\mu, \alpha \mathbf{I})$. Then, as shown in [2], the maximizing covariance matrix Q^o has a spectral decomposition $Q^o = U_\mu^o \Lambda^o U_\mu^{o\dagger}$, where the first column of the unitary matrix U_μ^o is given by $U_\mu^o[1] = \frac{\mu}{||\mu||}$, and $U_\mu^o[2] \cdots U_\mu^o[n_T]$ are arbitrarily chosen, except for the restriction that the columns of U_μ^o are orthonormal. Furthermore, the eigenvalues $\lambda_2^o = \cdots = \lambda_{min}^o \triangleq \lambda^o$, where $\lambda^o = \frac{P - \lambda_1^o}{n_T - 1}$.

C. Optimality of Beamforming and Quality of Feedback

- With perfect feedback, i.e. $\mathbf{h} \sim \tilde{N}(\mu, \mathbf{0})$, the capacity achieving input covariance matrix $Q^o = P\frac{\mu\mu^\dagger}{\mu^\dagger\mu}$ has rank one and therefore beamforming achieves capacity [4].
- With no feedback, i.e. $\mathbf{h} \sim \tilde{N}(\mathbf{0}, \alpha \mathbf{I})$, the optimum covariance matrix derived by Telatar in [1], $Q^o = \frac{P}{n_T}\mathbf{I}$, has full rank. So beamforming is not optimal.
- For the special case $\mathbf{h} \sim \tilde{N}(\mathbf{0}, \Sigma)$, it is shown in [3] that beamforming in the direction corresponding to the largest eigenvalue of the channel covariance matrix Σ is asymptotically optimum as the SNR tends to zero.
- Numerical results in [2] indicate that beamforming becomes the optimal strategy as the quality of feedback improves under mean feedback or if there is a strong channel mode present under covariance feedback.

IV. Results

Our main result is contained in the following Theorem.

Theorem 1: The Shannon capacity can be achieved with a unit rank matrix if and only if the following condition is true:

$$\mathrm{E}\left[\frac{1}{1 + \frac{P\lambda_1^\Sigma}{\sigma^2} w_1}\right] \le \frac{1}{1 + \frac{P\lambda_2^\Sigma}{\sigma^2}} \tag{2}$$

where, for covariance feedback

1. $\lambda_1^\Sigma > \lambda_2^\Sigma$ are the two largest eigenvalues of the channel covariance matrix Σ,
2. w_1 is exponential distributed with unit mean, i.e. $w_1 \sim e^{-w1}$,

and for mean feedback

1. $\lambda_1^\Sigma = \lambda_2^\Sigma = \alpha$, and

2. w_1 has a noncentral chi-squared distribution. More precisely $w_1 \sim e^{-\frac{||\mu||^2}{\alpha} - w_1} I_0\left(2||\mu||\sqrt{\frac{w_1}{\alpha}}\right)$ where $I_0(.)$ is the 0th-order modified Bessel function of the first kind.

Further, for covariance feedback the expectation can be evaluated to get the following closed form expression for the necessary and sufficient condition.

$$\frac{\sigma^2}{P\lambda_1^\Sigma} e^{\frac{\sigma^2}{P\lambda_1^\Sigma}} \Gamma(0, \frac{\sigma^2}{P\lambda_1^\Sigma}) \leq \frac{1}{1 + \frac{P\lambda_2^\Sigma}{\sigma^2}}. \tag{3}$$

Proof: We consider the case of covariance feedback first. From Section III-A we know that the eigenvectors of the optimum input covariance matrix are the columns of U_Σ. So we substitute $Q = U_\Sigma \Lambda_Q U_\Sigma^\dagger$ in (1) to obtain

$$C = \max_{\Lambda_Q \ : \ \sum_{i=1}^{n_T} \lambda_i^Q = P} \mathrm{E}\left[\log[1 + \frac{\mathbf{h}^\dagger U_\Sigma \Lambda_Q U_\Sigma^\dagger \mathbf{h}}{\sigma^2}]\right]. \tag{4}$$

Define $\mathbf{z} \triangleq \Lambda_\Sigma^{-\frac{1}{2}} U_\Sigma^\dagger \mathbf{h}$ so that $\mathbf{z}$ is white and the capacity can be expressed as

$$C = \max_{\Lambda_Q \ : \ \mathrm{trace}(\Lambda_Q) = P} \mathrm{E}\left[\log\left(1 + \frac{\mathbf{z}^\dagger \Lambda_\Sigma^{\frac{1}{2}} \Lambda_Q \Lambda_\Sigma^{\frac{1}{2}} \mathbf{z}}{\sigma^2}\right)\right] \tag{5}$$

$$= \max_{\Lambda_Q \ : \ \sum_{i=1}^{n_T} \lambda_i^Q = P} \mathrm{E}\left[\log(1 + \sum_{i=1}^{n_T} \frac{\lambda_i^\Sigma \lambda_i^Q}{\sigma^2} w_i)\right] \tag{6}$$

where $w_i = |z_i|^2, i \in \{1, 2, \cdots, n_T\}$ are unit mean i.i.d. random variables with an exponential distribution. Interestingly, this is the capacity with n_T independent mutually interfering Rayleigh fading channels to a single receive antenna. A total transmit power P is distributed so that λ_i^Q is the transmit power on the i^{th} channel. The average channel power gain of the i^{th} channel is λ_i^Σ, the i^{th} eigenvalue of the original channel covariance matrix.

Now the beamforming solution corresponds to Q being a rank one matrix. So the vector of eigenvalues of Q (the diagonal entries of Λ_Q) becomes $\{\lambda_i^Q\} = \{P, 0, \cdots, 0\}$ for beamforming. Instead, suppose we allocate power $P - p$ to the dominant eigenvector of Σ and distribute the remaining power p among the remaining eigenvectors so that $\lambda_i^Q = \alpha_i p, i \in \{2, 3, \cdots, n_T\}$, where α_is are all positive and add up to unity. The capacity with such a power allocation becomes

$$C(p) = \mathrm{E}\left[\log(1 + \frac{(P-p)\lambda_1^\Sigma}{\sigma^2} w_1 + \sum_{i=2}^{n_T} \frac{p\alpha_i \lambda_i^\Sigma}{\sigma^2} w_i)\right]. \tag{7}$$

Evaluating the derivative of this capacity with respect to p at the point $p = 0$ we obtain

$$\left.\frac{\partial C(p)}{\partial p}\right|_{p=0} = \mathrm{E}\left[\frac{\sum_{i=2}^{n_T} \frac{\alpha_i \lambda_i^\Sigma}{\sigma^2} w_i - \frac{\lambda_1^\Sigma}{\sigma^2} w_1}{1 + \frac{P\lambda_1^\Sigma}{\sigma^2} w_1}\right] \tag{8}$$

$$= \sum_{i=2}^{n_T} \frac{\alpha_i \lambda_i^\Sigma}{\sigma^2} \mathrm{E}[w_i] \mathrm{E}\left[\frac{1}{1 + \frac{P\lambda_1^\Sigma}{\sigma^2} w_1}\right] - \frac{1}{P}\left(1 - \mathrm{E}\left[\frac{1}{1 + \frac{P\lambda_1^\Sigma}{\sigma^2} w_1}\right]\right) \tag{9}$$

$$= \mathrm{E}\left[\frac{1}{1 + \frac{P\lambda_1^\Sigma}{\sigma^2} w_1}\right]\left(\sum_{i=2}^{n_T} \frac{\alpha_i \lambda_i^\Sigma}{\sigma^2} + \frac{1}{P}\right) - \frac{1}{P}. \tag{10}$$

For beamforming to be the optimum strategy we need $\left.\frac{\partial C(p)}{\partial p}\right|_{p=0} \leq 0$ for all positive α_i that add up to unity. However note that $\sum_{i=2}^{n_T} \frac{\alpha_i \lambda_i^\Sigma}{\sigma^2}$ is maximized if $\alpha_2 = 1$ and the rest of α_i are 0. So we could just allocate the power p to the next strongest eigenvector. Therefore the necessary condition for beamforming to be optimal is given as follows:

$$\mathrm{E}\left[\frac{1}{1+\frac{P\lambda_1^\Sigma}{\sigma^2}w_1}\right]\left(\frac{\lambda_2^\Sigma}{\sigma^2}+\frac{1}{P}\right)-\frac{1}{P} \leq 0 \tag{11}$$

$$\Rightarrow \mathrm{E}\left[\frac{1}{1+\frac{P\lambda_1^\Sigma}{\sigma^2}w_1}\right] \leq \frac{1}{1+\frac{P\lambda_2^\Sigma}{\sigma^2}} \tag{12}$$

$$\frac{\sigma^2}{P\lambda_1^\Sigma} e^{\frac{\sigma^2}{P\lambda_1^\Sigma}} \Gamma(0, \frac{\sigma^2}{P\lambda_1^\Sigma}) \leq \frac{1}{1+\frac{P\lambda_2^\Sigma}{\sigma^2}}, \tag{13}$$

where $\Gamma(.,.)$ is the incomplete Gamma function. Note that (13) was obtained using the fact that w_1 is exponential distributed with unit mean.

Now the condition $\left.\frac{\partial C}{\partial p}\right|_{p=0} \leq 0$ is only necessary for optimality of beamforming. In order to prove that it is also sufficient we show that the second derivative of capacity with respect to p, $\frac{\partial^2 C}{\partial p^2} < 0$ for all $p \in (0, P)$. This is easily seen as follows:

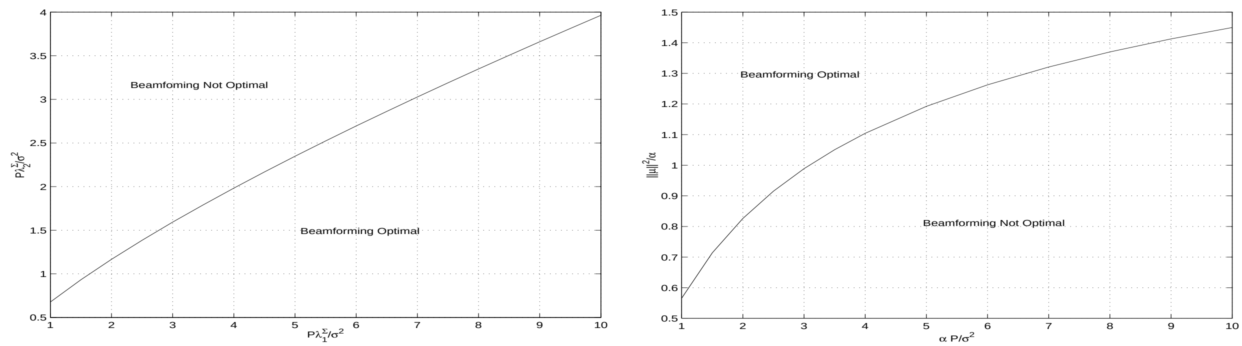
$$\frac{\partial^2 C}{\partial p^2} = -\mathrm{E}\left[\frac{\sum_{i=2}^{n_T}\frac{\alpha_i\lambda_i^\Sigma}{\sigma^2}w_i - \frac{\lambda_1^\Sigma}{\sigma^2}w_1}{1+\frac{(P-p)\lambda_1^\Sigma}{\sigma^2}w_1+\sum_{i=2}^{n_T}\frac{p\alpha_i\lambda_i^\Sigma}{\sigma^2}w_i}\right]^2 < 0. \tag{14}$$

Thus the necessary condition for optimality of beamforming is also sufficient.

In the case of mean feedback, the solution to the capacity optimization problem in Section III-B implies that for the optimal input covariance matrix to have rank one, $\lambda_1^o = P$ and $\lambda_2^o = \cdots = \lambda_{n_T}^o = 0$. The derivation of the necessary and sufficient condition in this case proceeds in a similar fashion to the covariance feedback case presented earlier. The only significant difference is that now $\mathbf{z}$ needs to be defined as $\mathbf{z} \triangleq \frac{1}{\sqrt{\alpha}} U_\mu^{o\dagger}\mathbf{h}$. ■

V. Discussion

The necessary and sufficient condition for optimality of beamforming for both mean and covariance feedback (2) is plotted in Figure 1. It clearly marks out the regions where inequality (2) is satisfied and beamforming is the optimal strategy and where the inequality is reversed and so capacity cannot be achieved with a unit rank input covariance matrix. For the covariance feedback case, note that the optimality of beamforming depends only on the channel SNR associated with the two dominant eigenvalues of the channel covariance matrix. Also note that as the SNR itself increases, greater disparity is required between the two strongest eigenvalues for optimality of beamforming. This is consistent with the notion of water pouring over the channel modes (eigenvalues). Beamforming involves using just the strongest channel mode. If the disparity between the two strongest (deepest) modes is high enough or the amount of water (total SNR) small enough, then the waterfill just covers the strongest mode. So if the second strongest mode is left unused the actual



(a) Covariance Feedback

(b) Mean Feedback

Fig. 1. Necessary and Sufficient Condition for Optimality of Beamforming

strength of the remaining modes (eigenvalues) does not matter as long as they are weaker (shallower). Hence the condition for optimality does not involve the remaining eigenvalues.

For mean feedback, the optimality of beamforming depends only upon the channel SNR $P\frac{\alpha}{\sigma^2}$ and the 'quality of feedback' $\frac{||\mu||^2}{\alpha}$. As mentioned earlier, for perfect feedback ($\frac{||\mu||^2}{\alpha} \rightarrow \infty$) the optimal strategy is beamforming, while for no feedback ($\frac{||\mu||^2}{\alpha} \rightarrow 0$) the optimal strategy cannot be beamforming. From Figure 1 we note that for a given channel SNR, as the quality of feedback improves beamforming becomes the optimal strategy. Higher SNR requires better feedback quality for optimality of beamforming. This has a similar waterpouring interpretation as described above.

VI. Conclusions

We determined the precise form of the necessary and sufficient condition for when the capacity in an n_T transmit, single receive antenna system is achieved by a unit rank input covariance matrix (beamforming) for both covariance feedback and mean feedback. Our result justifies the observations and numerical results in [2] and [3] where it was found that beamforming performs close to the optimal strategy as the quality of feedback improves under mean feedback or the disparity between the two strongest channel modes increases under covariance feedback. So we conclude that in a multiple transmit antenna system the quality of feedback is beneficial on two accounts. Firstly, as shown in earlier work [2], the capacity with partial CSIT is significantly higher. Secondly this higher capacity is achieved with a *lower* complexity since an improved feedback makes beamforming the optimal strategy.

References

[1] E. Telatar, "Capacity of Multi-antenna Gaussian Channels", AT&T-Bell Labs Internal Tech. Memo., June 1995.

[2] E. Visotsky, U. Madhow, "Space-Time Transmit Precoding with Imperfect Feedback", Proceedings of ISIT 2000.

[3] A. Narula, M. J. Lopez, M. D. Trott, G. W. Wornell, "Efficient Use of Side Information in Multiple-Antenna Data Transmission over Fading Channels", IEEE JSAC, Vol. 16, No. 8, October 1998.

[4] A Narula, M. D. Trott, G. W. Wornell, "Performance Limits of Coded Diversity Methods for Transmitter Antenna Arrays", IEEE Transactions on Information Theory, Vol. 45, No. 7, November 1999.

[5] A. Goldsmith and P. Varaiya, "Capacity of fading channels with channel side information", IEEE Transactions on Information Theory, Vol 43, pp. 1986-91992, November 1997.