Further Results in the Unified Analysis of Digital Communication Systems

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Abstract—Seymour Stein, in a 1964 paper entitled “Unified Analysis of Certain Coherent and Noncoherent Binary Communications Systems,” unified the theory for analyzing the bit error probability (BEP) of generalized BPSK and FSK communication systems. This unified theory is powerful because the results are truly general, the expressions for the BEP are functions of well studied transcendental functions (Marcum’s Q-function, etc.), and the analysis technique is ideal for digitally implemented demodulator analysis. This unified analysis requires that the demodulation phase reference signal and the matched filter output both be complex (bandpass) Gaussian random processes. Stein’s method can be extended to the analysis of the BEP of QPSK, staggered QPSK, and MSK communication systems. The resultant noisy reference BEP waterfall curves are presented. The numerical advantages of this technique and some practical results are discussed. In a parallel manner the BEP for QAM communication systems can be analyzed. This technique is numerically more intensive but is used to generate noisy reference BEP waterfall curves for 16 QAM and 64 QAM modulations. Unfortunately, few carrier synchronizers produce a complex Gaussian reference signal, but pragmatically many reference signals can be accurately approximated by a complex Gaussian at moderate to high SNR. A comparison between actual BEP performance and the approximate results are presented.

I. INTRODUCTION

SEYMOUR Stein in his 1964 paper [1] presented a unified analysis technique for evaluating the probability of error in binary communication systems. This paper unified the error probability performance expressions for generalized binary noncoherent FSK reception and generalized differentially coherent BPSK reception. The real utility of this analysis technique is the wide variety of situations for which it has validity. This analysis technique is valid for noisy phase references (if they are bandpass Gaussian), interfering signals, intersymbol interference, fading signals, filter crosstalk and dependent noise. Another important characteristic of this technique is the amenability to the analysis of digitally implemented demodulators. Since implementations of digital communication systems are increasingly using gate array and VLSI technology, the ability to analyze digital demodulator structures will continue to require attention. This technique unifies many types of analyses performed in communication system design and is widely utilized in practice.

This analysis technique has been widely used and extended since the first paper. Henry [2] used this analytical technique to compare the bit error probability (BEP) performance of DPSK and FSK demodulation in the presence of a frequency uncertainty. Maciejko [3] used the results in [1] for a Rayleigh-fading channel performance analysis. Haeb and Meyr [4] generalized Maciejko’s work on QPSK and MSK in the Rayleigh fading channel with a noisy phase reference. The goal of this paper is to further unify the analytical techniques which were derived from Stein’s original work [1] for demodulation with a noisy reference.

Stein [1], among others (the earliest work was [21]), utilized transform domain techniques to derive an expression for

\[ P \left[ |t_1|^2 < |t_2|^2 \right] \]

where \( t_1 \) and \( t_2 \) are two uncorrelated nonzero mean complex Gaussian vectors. This is equivalent to

\[ P[R_1 < R_2] \]

where \( R_1 \) and \( R_2 \) are uncorrelated Rician random variables. The probability in (1) or (2) was reduced to an expression involving the Marcum’s Q-function. This result is significant because the evaluation of Marcum’s Q-function has received considerable attention in the literature and many methods are available for rapid evaluation [5, and references therein]. Having derived this result, Stein demonstrated that FSK, DPSK, and generalized BPSK demodulator error probability can be reduced to the form of (2).

The utilization of Stein’s analysis technique provides an elegant unification method for the analysis of digital communication system performance with a noisy phase reference. The digital communication system to be analyzed must satisfy two constraints for theoretical correct usage of this technique. The first constraint is that the in-phase (I) and quadrature (Q) channel must be independently modulated. This is not a stringent constraint; many of the uncoded modulation schemes satisfy this requirement, mainly QPSK, staggered QPSK (SQPSK), MSK, and QAM. The only practical uncoded modulation which does not satisfy the constraint is 8 PSK. Secondly the phase reference used for demodulation must be a complex Gaussian random variable. This second constraint is much harder to satisfy. Phase estimation schemes are typically nonlinear in nature [6]–[14] and do not produce Gaussian random variables. This paper will demonstrate at moderate to high reference SNR that these nonlinear estimation techniques produce a BEP performance which is accurately approximated.
by a complex Gaussian phase reference model. So most practical communication systems can be approximately analyzed at moderate to high values of $E_b/N_0$ with this technique.

The analysis of digital communication system performance with a noisy phase reference is a problem with a rich history [7, among others] but the generalization of Stein’s analysis technique provides improved analytical methods for this problem. The noisy phase reference problem has been evaluated for most practical modulations [10, for example]. Typically, the analysis techniques utilized in this problem are numerically intensive and give results for symbol error probability. A less numerically intensive analysis technique applicable to MPSK modulation is presented in [11]. The generalization of Stein’s analysis has four distinct advantages over these techniques. First, an analysis of the QAM-noisy phase reference problem can be addressed in contrast to the results of [11]. Second, this technique produces an expression for the BEP in contrast to the symbol error probability of the typical noisy phase reference analysis. Also BEP is expressed in terms of Marcum’s $Q$-function, $I_0(x)$ and $e^x$ which are well understood (numerically and theoretically) transcendental functions. Finally, Stein’s method can be generalized to fading channels (Rician and Rayleigh) without additional computation. This final point is most significant and [22] is an example of this generalization.

This paper is organized as follows. Section II describes the channel and signal models. In Section III a review of Stein’s method is presented. In Section IV a unified probability of error analysis is formulated for QPSK, SQPSK, MSK, and QAM modulations. Section V briefly overviews some common phase estimation architectures. A comparison between actual performance and the approximate performance derived with a complex Gaussian model for several phase estimation architectures is presented. Finally, Section VI presents some practical results and the conclusions of the paper.

II. ANALYSIS MODELS

Because of the attractiveness of the digital implementation, the receiver block diagram seen in Fig. 1 is of most interest. For the wideband additive noise channel this receiver can be modeled by Fig. 2. The signal $z(t)$ represents the transmitted signal (in complex baseband form). The complex exponential multiplying $z(t)$ models the unknown phase induced by the channel. The noise is additive white Gaussian (AWG) and the receiver filter is matched to the modulation pulse shape. For the analysis performed in this paper, the pulse shapes for the nonstaggered modulations (BPSK, QPSK, and QAM) satisfy the Nyquist criterion for zero ISI, while the staggered modulations (MSK and SQPSK) have ideal pulse shapes (half cycle sinusoid and rectangular pulse shapes, respectively). The A/D converter samples the matched filter output and this sampled signal is used both to estimate the phase and demodulate the data sequence.

In this paper it is assumed that the symbol timing is known, the carrier frequency offset is negligible, and the phase process is a slowly varying random process. This is certainly not the case in general but the relaxation of these assumptions, although tractable [12], would cloud the major premise of this paper. That premise is that Stein’s analysis technique [1] provides the basis for a unified analysis technique for many digital modulations in the presence of a noisy phase reference and fading. It is shown in [12] that with these assumptions the sampled output of the matched filter for BPSK modulation will have the form

$$x(n) = a_n \sqrt{E_b} \exp[i \theta_0] + v(n)$$

where $n$ is the sample number, $a_n$ is the modulation symbol taking $\pm 1$ values ($a_n$ are i.i.d. for each sample), $E_b$ is the energy per bit, and $\theta_0$ is the unknown received phase angle. The noise sample $v(n)$ in equation (1) is a zero mean, delta correlated discrete time random process with a variance $N_0$. Likewise the sampled matched filter output for QPSK modulation will have the form

$$x(n) = (a_n + jb_n) \sqrt{E_b} \exp[i \theta_0] + v(n)$$

where $b_n$ is also a modulation symbol taking $\pm 1$ values.

Quadrature amplitude modulation (QAM) has a similar form to QPSK. The sampled output from the matched filter for 16 QAM has the form

$$x(n) = (a_n + jb_n) \sqrt{2E_b/5} \exp[i \theta_0] + v(n)$$

where $a_n$ and $b_n$ are the in-phase and quadrature modulation symbols. The symbols $a_n$ and $b_n$ are jointly independent and are individually i.i.d. sequences which take the values $\pm 1, \pm 3$

1 Since [12] might not be readily available, the matched filter output expression for MSK is developed in Appendix A to highlight the origin of the signal model presented in this section.
with equal probability. The matched filter output is sampled once per symbol for BPSK, QPSK, and 16 QAM.

The demodulation architecture typically implemented for these modulations can be seen in Fig. 3. This architecture uses the phase reference to demodulate the constellation and the symbol estimates are made using the real and imaginary parts of this demodulated signal. The phase reference used in the subsequent analysis has the form

$$r(n) = \sqrt{G} E_s \exp[j\theta_0] + \nu_2(n)$$  \hspace{1cm} (3)

where $G$ is the SNR gain\(^2\) of the reference signal [12] and $E_s$ is the symbol energy. For $r(n)$ to have this form the reference recovery subsystem is producing an unbiased estimate of $\theta_0$.

The results of this paper can be extended to the case of a biased phase estimator (this is done in [12] for BPSK and QPSK) but will not be done in this paper. The discrete noise process $\nu_2(n)$ also has a variance of $N_0$. This model of a reference signal can approximate that produced by a digital phase-locked loop (PLL) or a digital carrier synchronizer [8], [9], [14] for moderate to high values of reference SNR. For these reasons this architecture pragmatically models a wide range of communication receivers.

QPSK and MSK digital demodulators have different structures than the ones presented above since the in-phase and quadrature data symbols are staggered in time. The demodulation architectures for these modulations are seen in Fig. 4. For QPSK and MSK the matched output filter is sampled at the bit rate as compared to the symbol rate for QPSK and QAM. The signal $x(n)$ will for QPSK on even number samples have the form

$$x(2k) = \left[ a_{2k} + j \left( b_{2k-1} + b_{2k+1} \right) / 2 \right] \sqrt{E_s} \exp[j\theta_0] + \nu(2k)$$

and likewise on odd samples the form is

$$x(2k + 1) = \left[ a_{2k} + \frac{a_{2k+1} + a_{2k+1}}{2} + j b_{2k+1} \right] \sqrt{E_s} \exp[j\theta_0] + \nu(2k + 1).$$

These forms are similar to that of (2) except that the quadrature modulation can change during a symbol. The response of MSK to a matched filter is very similar to QPSK and the detailed analysis is presented in Appendix A. For MSK modulation the even numbered samples of $x(n)$ will have the form

$$x(2k) = \left[ a_{2k} + j \left( b_{2k-1} + b_{2k+1} \right) / \pi \right] \sqrt{E_s} \exp[j\theta_0] + \nu(2k)$$

and likewise odd samples will have the form

$$x(2k+1) = \left[ a_{2k+1} + \frac{a_{2k} + a_{2k+2}}{2} + j b_{2k+1} \right] \sqrt{E_s} \exp[j\theta_0] + \nu(2k+1).$$

In MSK the factor of $\pi$ results from the half cycle sinusoid matched filter not being time aligned with the I/Q pulse shape. The phase reference for the staggered modulations has the same form as (3). The analysis models presented in this section provide a simple and yet accurate representation of the signals in digital demodulators.

III. STEIN’S METHOD

Stein's method is a generalization of the idealized binary FSK BEP analysis. In idealized binary FSK the BEP can be expressed as

$$P(E) = P(R_1 < R_2)$$

where $R_1$ and $R_2$ are the envelope samples out of each filter (i.e., Mark and Space, etc.). These envelope samples are Rice variates (note: a Rician random variable results from taking the envelope of a complex Gaussian random variable). Stein, among others, found techniques to reduce this BEP expression to

$$P(R_1 < R_2) = \frac{1}{2} \left[ 1 - Q\left( \sqrt{b}, \sqrt{a} \right) + Q\left( \sqrt{a}, \sqrt{b} \right) \right] A \exp \left[ -\frac{(a + b)}{2} \right] J_0 \left( \sqrt{ab} \right)$$  \hspace{1cm} (6)

where

$$a = \frac{|E(R_1)|^2}{\sigma_1^2 + \sigma_2^2}$$

$$b = \frac{|E(R_2)|^2}{\sigma_1^2 + \sigma_2^2}$$

$$A = \frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

$$Q(\cdot) = \text{Marcum's Q-function}$$

$$J_0(\cdot) = \text{Modified Bessel function of order zero}$$

with $2\sigma^2 = \text{var}[R_1]$ and $2\sigma^2 = \text{var}[R_2]$.

Stein recognized a common identity in complex analysis enabling the application of the FSK result to a BPSK analysis. This identity is

$$\text{Re}[x(n)r^*(n)] = \frac{|x(n) + r(n)|^2}{4} - \frac{|x(n) - r(n)|^2}{4}$$  \hspace{1cm} (7)
Since the BEP for BPSK normally reduces to
\[ P(E) = P(\text{Re}[x(n)\ast(n)] \leq 0), \]
Stein used (7) to produce
\[ P(E) = P(R_1 < R_2) \]
where
\[ R_1 = \frac{|x(n) + r(n)|}{2} \quad R_2 = \frac{|x(n) - r(n)|}{2}. \]
This reduces the BPSK BEP analysis to an equivalent FSK analysis enabling the use of (6). Equation (6) demonstrates that the BEP of BPSK is dependent only on the first- and second-order moments of \( x(n) \) and \( r(n) \). For the Rician-fading channel these moments are
\[ |E[x(n)]| = \sqrt{\frac{K\overline{E}_b}{K+1}} \quad |E[r(n)]| = \sqrt{\frac{KG_r\overline{E}_b}{K+1}} \]
and
\[ \text{var}[x(n)] = N_0 + \frac{\overline{E}_b}{K+1} \quad \text{var}[r(n)] = N_0 + \frac{G_r\overline{E}_b}{K+1} \]
where \( K \) is the ratio between the specular and random signal power, \( G_s \) is the specular SNR gain of the reference signal, \( G_r \) is the random SNR gain of the reference signal and \( \overline{E}_b \) is the average energy per bit. Using these moments, the BPSK BEP for generalized Rician fading and a noisy phase reference can be expressed as (6) with
\[ a = \frac{1}{2} \left( \sqrt{\frac{K}{K+1}} - \sqrt{\frac{G_s\overline{E}_b}{1 + G_s\overline{E}_b}} \right) \]
\[ b = \frac{1}{2} \left( \sqrt{\frac{K}{K+1}} + \sqrt{\frac{G_s\overline{E}_b}{1 + G_s\overline{E}_b}} \right) \]
\[ A = \frac{E((x(n) - \overline{x})(r(n) - \overline{r}))^2}{2\sigma_x\sigma_r} \]
\[ = \text{correlation coefficient between the matched filter output and the reference signal} \]
and \( \overline{E}_b \) is the average \( E_b/N_0 \). This is a powerful result. One expression can be used to generate BPSK BEP analyses for fading channels, noisy phase references and a combination of both. This power and utility is demonstrated in Fig. 5 where plots of DPSK \((G = 1)\) and ideal BPSK \((G = \infty)\) are presented for the nonfading \((K = \infty)\), the slow Rician fading \((K = 10)\) and the slow Rayleigh-fading \((K = 0)\) channels. Slow in this case implies that the fading changes slowly in comparison to the memory length of the phase estimator. This implies that \( G_s = G_r = G \) and
\[ A = \frac{\overline{E}_b}{K+1} \left( \frac{\overline{E}_b}{K+1} + \frac{1}{G} \right) \]
The emphasis of this paper is the analysis of the BEP with a noisy phase reference in a nonfading channel. For this case \((K = \infty)\) (8) reduces to
\[ a = \frac{1}{2} \left( \sqrt{GR_b} - \sqrt{\overline{E}_b} \right)^2 \]
\[ b = \frac{1}{2} \left( \sqrt{GR_b} + \sqrt{\overline{E}_b} \right)^2 \]
\[ A = 0. \]
The BEP waterfall curves for various values of gain are seen in Fig. 6 (note that \( G = 1 \) - DPSK). This analysis technique is numerically much more efficient than the methods proposed in [10].

The proceeding results demonstrate the power of Stein’s analysis technique. The remainder of the paper will demonstrate how these concepts can be extended to the analysis of the BEP of other digital modulations.

IV. UNIFIED BEP ANALYSIS

The use of the analysis technique highlighted in Section III can be extended for use on QPSK, SOPSK, MSK, and OAM modulation performance analysis. The first step in this extension is expressing BEP for modulations for which the in-phase and quadrature components of the carrier are independently...
and identically modulated as
\[ P_B(E) = \frac{1}{2} P(E_1) + \frac{1}{2} P(E_Q) \]  \hspace{1cm} (9)
where \( P(E_1) \) and \( P(E_Q) \) are the probabilities of bit error for the in-phase and quadrature data, respectively. This result is derived in Appendix B. A trivial example of this result can be seen in the fact that the BEP of ideal BPSK and QPSK are identical. This result combined with Stein’s analysis technique produces expressions for the BEP probability of QPSK, SQPSK, and MSK.

A. QPSK Modulation

The use of (9) and Stein’s method greatly simplifies the BEP analysis of QPSK. The estimate of the in-phase and quadrature data symbols follows the form (see Fig. 3)
\[ a_n = \text{sgn} \left[ \text{Re} (x(n) r^*(n)) \right] \]
\[ b_n = \text{sgn} \left[ \text{Im} (x(n) r^*(n)) \right] \]
\[ = \text{sgn} \left[ \text{Re} \left( x(n) r^*(n) \exp \left( -j \frac{\pi}{2} \right) \right) \right] \]
Thus, both \( P(E_1) \) and \( P(E_Q) \) can be expressed in a form amenable to Stein’s method. Assuming, without loss of generality, that \( a_n = 1 \) and \( b_n = 1 \), the in-phase and quadrature BEP are
\[ P(E_1) = P(\text{Re} [x(n) r^*(n)] < 0) \]
\[ P(E_Q) = P \left( \text{Re} \left[ x(n) r^*(n) \exp \left( j \frac{\pi}{2} \right) \right] < 0 \right) \]
Using the results of [1] both \( P(E_1) \), \( P(E_Q) \), and the BEP can be expressed as
\[ P_B(E) = P(E_1) = P(E_Q) \]
\[ = \frac{1}{2} \left[ 1 - Q(\sqrt{r_3}, \sqrt{r_3}) + Q(\sqrt{r_4}, \sqrt{r_4}) \right] \]  \hspace{1cm} (10)
where
\[ r_3 = GR_6 + R_6 - R_6 \sqrt{2G} \]
\[ r_4 = GR_6 + R_6 + R_6 \sqrt{2G} \]

and \( R_6 = \frac{P_0}{N} \). The BEP waterfall curves for various values of reference SNR gain \( G \) are seen in Fig. 7. This method again greatly reduces the numerical burden of calculating the BEP from those proposed in [10]. An interesting result of this derivation is the BEP for DPSSK demodulation which can be expressed as (10) with \( G = 1 \), i.e., \( r_3 = R_6 (2 - \sqrt{2}) \), \( r_4 = R_6 (2 + \sqrt{2}) \).

B. SQPSK Modulation

The method simplifying the BEP calculations for SQPSK is very similar to that of QPSK. The estimates of the in-phase and quadrature data symbols have the form (see Fig. 4)
\[ a_{2k} = \text{sgn} \left[ \text{Re} (x(2k) r^*(2k)) \right] \]
\[ b_{2k+1} = \text{sgn} \left[ \text{Re} (x(2k+1) r^*(2k+1) \exp \left( -j \frac{\pi}{2} \right)) \right] \]  \hspace{1cm} (11a)
\[ b_{2k+1} = \text{sgn} \left[ \text{Re} (x(2k+1) r^*(2k) \exp \left( -j \frac{\pi}{2} \right)) \right] \]  \hspace{1cm} (11b)
In developing the equations for the SQPSK BEP we will consider only the in-phase data symbol estimates \( (a_{2k}) \). As in the case of QPSK, \( P(E_Q) \) reduces to identical expressions as \( P(E_1) \) for the model presented in this paper. Assuming without loss of generality that \( a_{2k} = 1 \) and using the results of Section II, the expression for \( x(2k) \) reduces to
\[ x(2k) = a_{2k} \sqrt{E_s} \exp \left( j \theta_0 \right) + \nu(2k) \]
when a transition in the \( Q \) data occurs \( (b_{2k+1} \neq b_{2k-1}) \) and
\[ x(2k) = [a_{2k} + j b_{2k+1}] \exp \left( j \theta_0 \right) + \nu(2k) \]
when no transition occurs. With this assumption an error now occurs when \( \text{Re} (x(2k) r^*(2k)) < 0 \) and \( P(E_1) \) can be calculated using Stein’s method. If the quadrature modulation symbols are equally probable i.i.d. random variables, then the BEP for the in-phase data can then be expressed via total probability as
\[ P_B(E) = \frac{1}{2} P(E_1|T) + \frac{1}{2} P(E_1|NT) \]  \hspace{1cm} (12)
where \( T \) indicates a transition occurs in the quadrature data and \( NT \) indicates no transition occurs in the quadrature data. When a transition occurs the BEP expression is identical to that of BPSK except the reference SNR is larger. Hence,
\[ P(E_1|T) = \frac{1}{2} \left[ 1 - Q(\sqrt{r_{s1}}, \sqrt{r_{s2}}) + Q(\sqrt{r_{s2}}, \sqrt{r_{s1}}) \right] \]  \hspace{1cm} (13a)
where
\[ r_{s1} = \frac{1}{2} \left( \sqrt{2GR_6} - \sqrt{R_6} \right)^2 \]  \hspace{1cm} (13b)
\[ r_{s2} = \frac{1}{2} \left( \sqrt{2GR_6} + \sqrt{R_6} \right)^2 \]  \hspace{1cm} (13c)
Combining the expressions in (13) and (14) as in (12) results in an expression for the BEP of SQPSK. The BEP waterfall curves for various values of reference SNR gain (G) are seen in Fig. 8. A comparison of Figs. 7 and 8 shows SQPSK is less sensitive to a noisy phase reference than QPSK as was discussed in [16].

C. MSK Modulation

The analysis of conventional MSK is very similar to the analysis of SQPSK. Both conventional MSK and SQPSK are modulations in which the I and Q data transitions do not occur at the same time. Since this is the case, the estimates of the in-phase and quadrature symbols have the same form as (11). Again $P(E_I)$ and $P(E_{Q})$ are identical due to the symmetry of the problem, hence, again only $P(E_I)$ will be calculated. This can be done by assuming $a_k = 1$ and realizing that

$$P(E_I) = P(\text{Re}[x(2k)e^{\ast}(2k)] < 0).$$

(15)

Again $x(2k)$ will take two different forms depending on whether a transition occurs in the quadrature channel. Again using the MSK results from Section II, $x(2k)$ has the form

$$x(2k) = \left( a_{2k} + jb_{2k} \frac{2}{\pi} \right) \sqrt{E_b} \exp(j\theta_0) + \nu(2k)$$

when no transition occurs and the form

$$x(2k) = a_{2k} \sqrt{E_b} \exp(j\theta_0) + \nu(2k)$$

when a transition occurs. With $x(2k)$ having the above form (15) is amenable to Stein’s analysis and the form of the BEP of MSK is reduced to a simpler expression. The probability of error when a transition occurs ($P(E_I|T)$) is identical to (13). $P(E_I|NT)$ can be expressed as

$$P(E_I|NT) = \frac{1}{2} \left[ 1 - Q(\sqrt{r_{m1}}, \sqrt{r_{m1}}) + Q(\sqrt{r_{m2}}, \sqrt{r_{m2}}) \right]$$

(16a)

where

$$r_{m1} = GR_b + \frac{\pi^2 + 4}{2\pi^2} R_b - \sqrt{2G} R_b$$

(16b)

$$r_{m2} = GR_b + \frac{\pi^2 + 4}{2\pi^2} R_b + \sqrt{2G} R_b.$$  

(16c)

Again assuming a 50% transition density, (13) and (16) can be used to calculate $P(E_I)$ via

$$P(E_I) = P_{II} = \frac{1}{2} P(E_I|NT) + \frac{1}{2} P(E_I|T).$$

The corresponding waterfall curves for various values of gain are seen in Fig. 9.

It is interesting to note the difference between the analysis of SQPSK and conventional MSK. The only difference is the signal associated with the quadrature modulation. Since the MSK pulse is not constant over the symbol time, the sample out of the matched filter will not be as large. The signal amplitude for the quadrature modulation is $\sqrt{E_b}$ for SQPSK while it is $\frac{\sqrt{E_b}}{2}$ for MSK. This is important since a large portion of the BEP degradation due to a noisy phase reference in quadrature modulations can be attributed to crosstalk from...
the quadrature signal. This implies MSK is less sensitive to a noisy phase reference than SOPSK. This can be verified by comparing Figs. 8 and 9. A demodulator for serial MSK (FFSK) can have an identical structure with the exception that the data is differentially decoded.

D. QAM Modulation

The BEP for 16 QAM can be developed with similar methods as previously presented. Again using (9) and the symmetry of the problem the BEP for the demodulation of 16 QAM as presented in this paper is

\[ P_B(E) = P(E_1). \]

This probability can be expressed as [10]

\[
P(E_1) = \frac{1}{4} P(E_1|a_n = 1, b_n = 1) \]
\[
+ \frac{1}{4} P(E_1|a_n = 3, b_n = 1) \]
\[
+ \frac{1}{4} P(E_1|a_n = 1, b_n = 3) \]
\[
+ \frac{1}{4} P(E_1|a_n = 3, b_n = 3). \quad (17)
\]

The BEP will be determined by the form of the \( I \) and \( Q \) data slicers in Fig. 3. The typical form of the in-phase slicer

For 16 QAM is

\[
\hat{a}_n = 3 \quad \text{if} \quad \text{Re}[x(n)r^*(n)] > \sqrt{\frac{32}{5} E_b} \\
\hat{a}_n = 1 \quad \text{if} \quad 0 < \text{Re}[x(n)r^*(n)] \leq \sqrt{\frac{32}{5} E_b} \\
\hat{a}_n = -1 \quad \text{if} \quad -\sqrt{\frac{32}{5} E_b} < \text{Re}[x(n)r^*(n)] \leq 0 \\
\hat{a}_n = -3 \quad \text{if} \quad \text{Re}[x(n)r^*(n)] \leq -\sqrt{\frac{32}{5} E_b}.
\]

The quadrature slicer will have a similar form. Using this form for the data slicer, (17) can be evaluated by considering each term individually. Since the typical practice for QAM modulations is Grey encoding both the \( I \) and \( Q \) modulations, this fact is assumed. The first term of (17) now is expressed as

\[
P(E_1|a_n = 1, b_n = 1) = \frac{1}{2} P[\hat{a}_n = 3|a_n = 1, b_n = 1] \\
+ \frac{1}{2} P[\hat{a}_n = 1|a_n = 1, b_n = 1] \\
+ \frac{1}{2} P[\hat{a}_n = 3|a_n = 1, b_n = 1] \\
= \frac{1}{2} P \left[ \text{Re}[x(n)r^*(n)] > \sqrt{\frac{32}{5} E_b} \right] \\
+ \frac{1}{2} P \left[ \text{Re}[x(n)r^*(n)] < 0 \right] \\
+ \frac{1}{2} P \left[ \text{Re}[x(n)r^*(n)] < -\sqrt{\frac{32}{5} E_b} \right].
\]

Using (7) this result is written as

\[
P(E_1|a_n = 1, b_n = 1) = \frac{1}{2} P \left[ R_1^2 - R_2^2 > \sqrt{\frac{32}{5} E_b} \right] \\
+ \frac{1}{2} P \left[ R_1^2 < R_2^2 \right] \\
+ \frac{1}{2} P \left[ R_1^2 - R_2^2 < -\sqrt{\frac{32}{5} E_b} \right] \quad (18)
\]

where \( R_1 \) and \( R_2 \) are independent Rician variables. The second term can be reduced to the form of (6) but a similar reduction of the other two terms was not apparent to the author. Defining

\[
F_4(a, b, y) = \int_0^\infty \int_0^\infty f(R_1) dR_1 f(R_2) dR_2 \\
= \int_0^\infty \int_0^\infty R_1 \exp \left[ -\frac{a^2 + R_1^2}{2} \right] \\
\cdot I_0(aR_1) dR_1 f(R_2) dR_2 \\
= \int_0^\infty \left[ 1 - Q \left( a, \sqrt{R_1^2 + y} \right) \right] \\
\cdot R_2 \exp \left[ -\frac{b^2 + R_2^2}{2} \right] I_0(bR_2) dR_2.
\]
(18) can be expressed in a more compact form. This form is

\begin{align}
P[E_1|a_n = 1, b_n = 1] &= 1 - \frac{1}{2} F_s(\sqrt{q_1}, \sqrt{q_2}, y_1) \\
&\quad - \frac{1}{2} Q\left(\sqrt{\frac{q_1}{2}}, \sqrt{\frac{q_2}{2}}\right) \\
&\quad + \frac{1}{2} Q\left(\sqrt{\frac{q_2}{2}}, \sqrt{\frac{q_1}{2}}\right) \\
&\quad + \frac{1}{2} F_s(\sqrt{q_1}, \sqrt{q_2}, -y_1) \quad (18a)
\end{align}

where

\begin{align}
q_1 &= \frac{4R_b}{5} + 4R_b \sqrt{\frac{2G}{5}} + 4GR_b \quad (18b) \\
q_2 &= 4R_b - 4R_b \sqrt{\frac{2G}{5}} + 4GR_b \quad (18c) \\
y_1 &= 32R_b \sqrt{\frac{G}{10}}. \quad (18d)
\end{align}

The terms expressed with the \( F_s(a, b, y) \) correspond to the decision boundaries which are not at the origin. The terms expressed with the \( Q(a, b) \) are derived by Stein technique and correspond to the decision boundary at the origin. Although the numerical evaluation of \( F_s(a, b, y) \) involves a double integral, the actual computation can be reduced in complexity since the inner integral can be expressed as a \( Q \)-function. Recursions have been developed to reduce the computation burdens associated with evaluating the \( Q \)-function [5]. The QAM BEP analysis problem can be formulated in a parallel manner as Stein's original work but an "elegant" solution has eluded the author.

An expression for the first term of (17) has been derived and the other three terms can be derived in a similar fashion. For the sake of brevity these results will be presented without justification.

\begin{align}
P[E_1|a_n = 3, b_n = 1] &= 1 - \frac{1}{2} F_s(\sqrt{q_3}, \sqrt{q_4}, y_1) \\
&\quad - \frac{1}{2} Q\left(\sqrt{\frac{q_3}{2}}, \sqrt{\frac{q_4}{2}}\right) \\
&\quad + \frac{1}{2} Q\left(\sqrt{\frac{q_4}{2}}, \sqrt{\frac{q_3}{2}}\right) \\
&\quad + \frac{1}{2} F_s(\sqrt{q_3}, \sqrt{q_4}, -y_1) \quad (18a)
\end{align}

where

\begin{align}
q_3 &= 4R_b + 4R_b \sqrt{\frac{2G}{5}} + 4GR_b \\
q_4 &= 4R_b - 4R_b \sqrt{\frac{2G}{5}} + 4GR_b
\end{align}

and finally

\begin{align}
P[E_1|a_n = 3, b_n = 3] &= 1 - \frac{1}{2} F_s(\sqrt{q_7}, \sqrt{q_8}, y_1) \\
&\quad - \frac{1}{2} Q\left(\sqrt{\frac{q_7}{2}}, \sqrt{\frac{q_8}{2}}\right) \\
&\quad + \frac{1}{2} Q\left(\sqrt{\frac{q_8}{2}}, \sqrt{\frac{q_7}{2}}\right) \\
&\quad + \frac{1}{2} F_s(\sqrt{q_7}, \sqrt{q_8}, -y_1) \quad (18a)
\end{align}

where

\begin{align}
q_7 &= \frac{3q}{5} R_b + 4R_b \sqrt{\frac{2G}{5}} + 4GR_b \\
q_8 &= \frac{3q}{5} R_b - 4R_b \sqrt{\frac{2G}{5}} + 4GR_b
\end{align}

with \( y_1 \) in all three of the above equations equal to \( y_1 \) in (18). These four expressions can now be combined via (17) to evaluate the BEP of 16 QAM. The BEP waterfall curves for various values of reference SNR gain \( G \) are seen in Fig. 10.

A similar analysis of 64 QAM is accomplished using the same technique (except 16 constellation points are considered instead of 4). The result is tedious and does not add further insight into the problem so the details will not be presented, but the resulting BEP waterfall curves for various values of reference SNR gain \( G \) are seen in Fig. 11.

V. PRAGMATIC NOISY PHASE REFERENCE ANALYSIS

For completely rigorous performance analysis of digital demodulators, the complex Gaussian model for a phase reference is not widely applicable. Most phase estimation architectures produce a phase estimate which is a nonlinear function of an input Gaussian process (typically the matched filter output) hence not a complex Gaussian random process. While several notable exceptions exist whose resulting phase reference is complex Gaussian (e.g., the ML phase estimate of an unmodulated sinusoid with known frequency [8] and the linear filtered transmitted reference [17], [22]), the majority of the practical digital phase estimates cannot rigorously be modeled as a complex Gaussian random variable. Pragmatically though, many carrier synchronization architectures can be accurately approximated as such for moderate to high reference SNR.
and linear filtering of the resultant I/Q signals. If the filtering is sufficiently narrow-band then the output will, by the central limit theorem, approach that of a complex Gaussian random process. This is especially true at moderate to high input SNR. Hence, the analytical techniques derived in Section IV can accurately approximate the resulting BEP performance.

Two examples will be considered comparing the BEP performance calculated for these ML-based digital phase estimators. First we will consider the decision-directed exponentially weighted phase estimator first proposed in [14]. This is the simplest of the decision-directed “open loop” digital phase estimators. A rigorous nonlinear analysis of this demodulator was accomplished using Markov chain theory and presented in [12], [15]. Fig. 12 presents the BEP waterfall curves resulting from the rigorous analysis and the analysis accomplished in Section IV for BPSK modulation and $\beta = 0.875$ ($G = 15$).

A similar comparison can be done for the nonlinear phase estimation architecture presented in [9] with QPSK modulation. Unfortunately, two additional factors must be considered for a proper comparison. First, the nonlinearity produces a loss in SNR (e.g., “squaring” loss). Using results in [9] for $F(\rho) = \rho^4$, this loss is

$$\rho_L = \frac{1}{1 + \frac{9}{4R_0} + \frac{3}{2R_0^2} + \frac{3}{16R_0^4}}.$$  

Second, a phase ambiguity exists with this nonlinear phase estimation technique which typically is resolved by differential encoding. The BEP of differential encoding for a slowly varying reference is given by

$$P_{BE}(E) = 2P(E)(1 - P(E)).$$

The use of these two results in conjunction with the results of Section IV provides an approximate performance analysis. This approximate analysis can be compared to the simulated BEP of the demodulator proposed in [9]. The comparison is for 15 taps in the sliding window accumulator smoothing filter ($G = 15\rho_L$) and the results are seen in Fig. 13. Examination of Figs. 12–13 demonstrates that the analysis technique presented in the last section can be used to accurately estimate the BEP of ML-based digital phase synchronizers and demodulators.

B. Phase-Locked Loop Based Digital Phase Estimation

The most common method of carrier synchronization utilizes a phase-locked loop (PLL) [7], [8]. Two observations make the results of Section IV accurate approximations to the BEP of PLL-based demodulators: 1) the Tikhonov density (characteristic of the PLL phase error [6], [7]) converges to the density of the phase of a complex Gaussian (this was recognized in [19]) and 2) the BEP of MPSK, QPSK, and MSK are functions only of the phase error produced by a noisy phase reference. So even though a PLL derived reference has a constant amplitude, the complex Gaussian phase reference model can be used to accurately approximate the BEP for these modulations. A similar comparison between the BEP of a PLL generated phase reference and the results predicted by the complex Gaussian model of the phase reference can be

A. ML Based Digital Phase Estimation

Several digital phase synchronization architectures have been proposed based on the ML estimation of carrier phase [8], [18]. The majority of the practical realizations have been derived from the low SNR asymptotic structure [9], [14] (nonlinear phase estimation) or the high SNR asymptotic structure [4], [14] (decision-directed phase estimation). The premise of these ML-based phase estimation structures is a nonlinear transformation of the matched filter output sample...
Fig. 12. BEP performance comparison between a Markov analysis [15] and approximation by a complex Gaussian for a reference signal generated by a decision-directed exponentially weighted digital carrier synchronizer. BPSK modulation, $\beta = 0.875$ ($G = 15$).

Fig. 13. BEP performance comparison between simulated results and the complex Gaussian approximation for the nonlinear digital phase estimator [9]. QPSK modulation, $F(\rho) = \rho^4$, and 15 taps ($G = 15p_{4}$).

Fig. 14. BEP comparison of the Tikhonov and complex Gaussian phase reference models. QPSK modulation, $B_L T = 0.01$ ($G = 10$).

developed. A sampled analog PLL ($B_L T = 0.01$)\(^4\) with an input signal of equal power to that of the modulated signal will be used to generate the phase reference. The modulation for the comparison will be QPSK and the conditional BEP given a phase error $\phi$ is given by [20]

$$P(E|\phi) = \frac{1}{4} \text{erfc}(\sqrt{R_b}(\cos \phi + \sin \phi)) + \frac{1}{4} \text{erfc}(\sqrt{R_b}(\cos \phi - \sin \phi)).$$

Integrating this function over the Tikhonov density provides the desired BEP. A comparison of this result and the approximate analysis of Section IV ($G = 10$) is seen in Fig. 14. Although the PLL is typically preceded by a nonlinearity (e.g., Costas loop), in practical systems (moderate to high loop SNR) the resultant phase error pdf typically still has a Tikhonov characteristic [7]. So by accounting for the SNR loss produced by the nonlinearity in a similar manner as above, the approximate analytical results of Section IV will match the system performance of PLL-based synchronizers well at moderate to high loop SNR.

The BEP of QAM will be effected by the amplitude of a noisy phase reference. So the complex Gaussian model would not accurately model a PLL based reference signal if the amplitude variations are significant. But two factors permit the complex Gaussian model to accurately model the PLL synchronization of a QAM demodulator. First, QAM requires a higher $E_b/N_0$ to achieve a given $P(E)$ than the other modulations, so the amplitude variations will be small in the moderate to high SNR regions. Second, the amplitude variations of the complex Gaussian model will actually produce a higher BEP performance. Hence the analytical technique of Section IV will provide an accurate upper bound to the actual performance with a PLL-based phase synchronizer.

VI. CONCLUSION

This paper has presented a unified method of analyzing the BEP of several commonly used uncoded modulations. The modulations of BPSK, QPSK, SOPSK, MSK, 16 QAM, and 64 QAM are analyzed. The purpose of the paper is to quantify the effects of a noisy complex Gaussian phase reference on the BEP performance in the AWGN channel. The results of this paper provide an efficient and accurate analytical tool. These analytical methods are numerically less intensive than previously published methods because of the simplifying analysis due to Stein and frequent usage of the Marcum $Q$-function. Marcum’s $Q$-function is numerically well studied. A second

\(^4\)For a PLL the SNR gain is given by $G = \frac{1}{B_L T}$.
TABLE I
THE REQUIRED REFERENCE SNR GAIN FOR VARIOUS DEGRADATIONS,
MODULATIONS, AND BIT ERROR PROBABILITIES

<table>
<thead>
<tr>
<th>Modulation</th>
<th>$P_{th}(E)$</th>
<th>$R_k$ for ideal $P_{th}(E)$</th>
<th>$G \leq 0.1$ dB</th>
<th>$G \geq 0.5$ dB</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPSK</td>
<td>$10^{-5}$</td>
<td>9.59</td>
<td>2.8</td>
<td>1.2</td>
</tr>
<tr>
<td>BPSK</td>
<td>$2 \times 10^{-2}$</td>
<td>3.25</td>
<td>8.4</td>
<td>2.2</td>
</tr>
<tr>
<td>QPSK</td>
<td>$10^{-5}$</td>
<td>9.59</td>
<td>22.6</td>
<td>4.7</td>
</tr>
<tr>
<td>QPSK</td>
<td>$2 \times 10^{-2}$</td>
<td>3.25</td>
<td>24.6</td>
<td>5.3</td>
</tr>
<tr>
<td>SQPSK</td>
<td>$10^{-5}$</td>
<td>9.59</td>
<td>12.8</td>
<td>3.2</td>
</tr>
<tr>
<td>SQPSK</td>
<td>$2 \times 10^{-2}$</td>
<td>3.25</td>
<td>14.8</td>
<td>3.3</td>
</tr>
<tr>
<td>MSK</td>
<td>$10^{-5}$</td>
<td>9.59</td>
<td>6.1</td>
<td>1.6</td>
</tr>
<tr>
<td>MSK</td>
<td>$2 \times 10^{-2}$</td>
<td>3.25</td>
<td>8.9</td>
<td>2.0</td>
</tr>
<tr>
<td>16 QAM</td>
<td>$10^{-5}$</td>
<td>13.44</td>
<td>32.3</td>
<td>7.0</td>
</tr>
<tr>
<td>16 QAM</td>
<td>$2 \times 10^{-2}$</td>
<td>6.69</td>
<td>34.2</td>
<td>6.6</td>
</tr>
<tr>
<td>64 QAM</td>
<td>$10^{-5}$</td>
<td>17.78</td>
<td>41.8</td>
<td>8.3</td>
</tr>
<tr>
<td>64 QAM</td>
<td>$2 \times 10^{-2}$</td>
<td>10.65</td>
<td>37.4</td>
<td>7.3</td>
</tr>
</tbody>
</table>

advantageous feature is that the analysis results in expressions for the BEP in contrast to the symbol error probability. Since BEP is typically the overall system performance metric, quantifying BEP is practically more important than the symbol error probability (this becomes a more significant concern at higher error probabilities typical of coded systems).

The results of this paper are also aesthetically pleasing. The analysis of the BEP performance of several common modulation schemes is accomplished through one unifying analytical structure. The results presented here provide an intuitive understanding of the noisy phase reference problem. The analytical results presented do not accurately reflect actual performance since the reference signal in practice is rarely a complex Gaussian. But these techniques and results are appropriate for first-order approximations of the BEP. The real advantage is the analytical results are derived in seconds of computer time while rigorous analysis or simulation requires hours of computer time. Finally, these results can be generalized to the fading channel. This generalization further unifies many analytical results.

Finally, I would like to present some practical results. In many communication systems the reference gain (i.e., PLL loop bandwidth) is derived by a tradeoff between reducing reference noise due to oscillator phase noise and AWGN. A smaller $G$ increases the AWGN component while decreasing the phase noise contribution. Since the analytical method presented here can rapidly calculate the effects of a noisy phase reference, I decided to use it to calculate the required gain to achieve certain degradations from ideal due solely to AWGN. The degradations I chose were 0.1 dB and 0.5 dB since they seem to appear most often in system specifications. I calculated the required gain at $P_{th}(E) = 2 \times 10^{-2}$ (typical of coded systems) and $P_{th}(E) = 10^{-5}$ (typical of uncoded systems). These results are presented in Table I.

**APPENDIX A**

MSK modulation can be characterized as having a half sinusoid pulse shape given by

$$p(t) = \sqrt{\frac{2}{T}} \sin \left( \frac{\pi t}{T} \right) \quad 0 \leq t \leq T$$

$$= 0 \quad \text{elsewhere}$$

where $T$ is the baud or symbol time. The $I$ and $Q$ modulations are staggered in time by $\frac{T}{2}$ seconds. The complex baseband model of the received signal in the simple channel considered in this paper is given by

$$y(t) = E_b \exp[j \theta_0] \sum_{i=-\infty}^{\infty} a_i p(t - iT - \varepsilon)$$

$$+ j b_i p \left( t - \frac{2i - 1}{2} T - \varepsilon \right) + \nu(t)$$

where $\nu(t)$ is a complex AWGN process of variance $N_0$. The matched filter output has the form

$$x(t) = \sqrt{\frac{2}{T}} \int_{-T}^{T} y(t') \sin \left( \frac{\pi t'}{T} \right) dt'$$

and it will be sampled every $\frac{T}{2}$ seconds to provide the sufficient statistics for demodulation. The 2kth sample will have the form

$$x(2k) = a_{2k} \sqrt{E_b} \exp[j \theta_0] \int_0^{T/2} 2 \sin^2 \left( \frac{\pi t}{T} \right) dt'$$

$$+ j b_{2k-1} \sqrt{E_b} \exp[j \theta_0] \int_0^{T/2} 2 \sin \left( \frac{\pi t}{T} \right) \cos \left( \frac{\pi t}{T} \right) dt'$$

$$+ j b_{2k-1} \sqrt{E_b} \exp[j \theta_0] \int_{T/2}^{T} -2 \sin \left( \frac{\pi t}{T} \right) \cos \left( \frac{\pi t}{T} \right) dt'$$

$$+ \nu(2k).$$

This reduces to

$$x(2k) = \left[ a_{2k} + j \left( b_{2k-1} + b_{2k+1} \right) \right] \sqrt{E_b} \exp[j \theta_0] + \nu(2k).$$

**APPENDIX B**

The probability of bit error for uncoded modulation when $M$ bits per symbol are transmitted can be expressed as

$$P_B(E) = \frac{1}{M} \sum_{i=1}^{M} i P[i \text{ bit error/symbol}].$$

Assuming $N_I$ bits are modulated on the in-phase portion of the carrier and $N_Q$ bits are independently modulated on the quadrature portion and since the set of error events is a disjoint set this BEP can be rewritten as

$$P_B(E) = \frac{1}{M} \sum_{j=0}^{N_I} \sum_{k=0}^{N_Q} (j + k) P[j \in I \cap k \in Q].$$
where $P[j E_i \cap k E_Q]$ is the probability of $j$ errors in the in-phase data and $k$ errors in the quadrature data. Reducing this expression further gives

$$
P_{21}(E) = \frac{1}{M} \sum_{j=0}^{N_I} \sum_{k=0}^{N_Q} P[j E_i \cap k E_Q]
+ \frac{1}{M} \sum_{k=0}^{N_Q} \sum_{j=0}^{N_I} P[j E_i \cap k E_Q].$$

By the use of total probability this reduces to

$$
P_{21}(E) = \frac{1}{M} \sum_{j=0}^{N_I} j P[j E_i] + \frac{1}{M} \sum_{k=0}^{N_Q} k P[k E_Q]
= \frac{N_I}{M} \left[ \frac{1}{N_I} \sum_{j=0}^{N_I} j P[j E_i] \right]
+ \frac{N_Q}{M} \left[ \frac{1}{N_Q} \sum_{k=0}^{N_Q} k P[k E_Q] \right]
= \frac{N_I}{M} P[E_I] + \frac{N_Q}{M} P[E_Q].$$

where $P[E_i]$ and $P[E_Q]$ are the probabilities of bit error for the in-phase and quadrature data, respectively. When the number of bits on the in-phase and quadrature channel are identical this reduces to

$$
P_{21}(E) = \frac{1}{2} P[E_I] + \frac{1}{2} P[E_Q].$$

ACKNOWLEDGMENT

I would like to thank Dr. W. C. Lindsey for pointing out [19] and taking the time to review the original manuscript. His comments and those of the reviewers made this manuscript a better piece of work. Also I am grateful to TRW Inc. who provided financial support during the preliminary portion of this work in the form of a doctoral fellowship.

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